

A low order extension the Liénard–Wiechert retardation equations to include the Thomas precession

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In a calculation that directly parallels the derivation of the Thomas precession, the first time derivative of the retarded potentials is derived. The solutions have to be integrated in time to obtain the potential solution.

The Thomas precession vanishes when the acceleration and velocity vectors are parallel, causing the solution for the dipole antenna to be the same as for the Liénard–Wiechert solution, and those solutions are in turn always solutions to the Maxwell equations. The solution for the current loop antenna is not a solution to the Maxwell equations. Field equations are obtained by restructuring the Proca equations that are commensurate with the low order retardation solutions. The solutions are not in the Lorentz gauge and they are not solutions to the unmodified Proca equations.

The high order terms are not solutions to the equations. In representing angular relationships, an argument is developed that derivatives beyond the first will be required for more complete solutions. The calculations are not in tensor form, but the tensors represent angular relationships, and the inference is based on the tensor irreducibility theorem. In being linear equations expressing angular relationships, the theorem implies that exact retardation equations do not exist unless the contravariant tensor of rank $n + 1$ is reducible.

I. INTRODUCTION

If a Lorentz transform is performed to the frame of reference of an accelerated particle at time t , followed by an infinitesimal transform to the velocity of the particle at time $t + dt$, the result is the same as a direct transform at time $t + dt$, followed by a space rotation^{1,5}. The Lorentz transform is a vector equation. A Lorentz transform followed by a space rotation is not. In not being representable with conventional vector relationships, the space is not flat.

The basis of the precession is that the Lorentz transform does not achieve closure after transforming full circle through three frames of reference. The coordinates in the last frame of reference appear to be rotated, or rotating if multiple infinitesimal transforms are performed in the frame of reference of a particle in a circular orbit.

One way of interpreting the relationships is that we should adopt the conclusions of the observer in the other frame of reference as our own – to see ourselves as others see us. That is because the other frame of reference could be our frame of reference next time. The perspective would not be acceptable if the coordinates in our frame of reference were spinning, but the retarded potentials are first-known in the frame of reference of the particle, so there is no conflict between the perspectives for potential equations. The potentials transform in the same way as the coordinates, so the rotation also affects the vector potential.

II. THE LIÉNARD–WIECHERT EQUATIONS

The Liénard–Wiechert (LW) retardation equations were obtained in the years 1898 and 1899. They remain the only known retardation equations. The following calculations appear to represent the next term of the same retardation series, meaning that they are compatible with the LW equations and their methods of analysis. The velocity of conduction electrons in stationary copper wire is so low that the first term of the series is the only one that is ever needed in those configurations, so these solutions do not replace the LW solutions in most applications. The LW equations are⁶

$$\begin{aligned} \mathbf{A}_{\text{LW}} &= q\mathbf{v}/[rc(1 + \hat{\mathbf{r}} \cdot \mathbf{v}/c)] \\ \psi_{\text{LW}} &= q/[r(1 + \hat{\mathbf{r}} \cdot \mathbf{v}/c)]. \end{aligned} \quad (1)$$

The vector \mathbf{r} points from the field point to the source, and \mathbf{v} is the retarded velocity of the particle. The electromagnetic fields are obtained from the retardation solutions with the equations

$$\mathbf{E} = -\nabla\psi - (\partial\mathbf{A}/\partial t)/c \quad (2)$$

$$\mathbf{H} = \nabla \times \mathbf{A}. \quad (3)$$

The first time derivative of the equations will be needed in the calculations. It is convenient to make the substitutions $r = (\mathbf{r} \cdot \mathbf{r})^{\frac{1}{2}}$ and $\hat{\mathbf{r}} = \mathbf{r}/((\mathbf{r} \cdot \mathbf{r})^{\frac{1}{2}})$ before performing the differentiations. After differentiating with respect to the time at the source, the substitutions $d\mathbf{r}/dt_S = \mathbf{v}$ and $d\mathbf{v}/dt_S = \mathbf{a}$ are made, where \mathbf{a} is the retarded acceleration. The solution is simpler if \mathbf{r} is converted back to the product of a magnitude and a unit vector as the last

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step. The time derivatives become

$$\begin{aligned}
d\mathbf{A}_{\text{LW}}/dt_S &= q\mathbf{a}/[rc(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)] \\
&\quad - q\mathbf{v}\mathbf{a}\cdot\hat{\mathbf{r}}/[rc^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2] \\
&\quad - qv^2\mathbf{v}/[r^2c^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2] \\
&\quad + q\mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/[r^2c^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2] \\
&\quad - q\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/[r^2c(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)] \\
d\psi_{\text{LW}}/dt_S &= -q\mathbf{a}\cdot\hat{\mathbf{r}}/[rc(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2] \\
&\quad - q\hat{\mathbf{r}}\cdot\mathbf{v}/[r^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)] \\
&\quad - qv^2/[r^2c(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2] \\
&\quad + q(\hat{\mathbf{r}}\cdot\mathbf{v})^2/[r^2c(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2].
\end{aligned} \tag{4}$$

The equations must be parameterized by the time at the field point in order for them to be usable as retardation equations. The connection is $t_S = t - (\mathbf{r}\cdot\mathbf{r})^{1/2}/c$. After differentiating with respect to t and selectively substituting $\mathbf{r} = r\hat{\mathbf{r}}$, the equation becomes $dt_S/dt = 1 - \hat{\mathbf{r}}\cdot(d\mathbf{r}/dt)/c$. But $d\mathbf{r}/dt$ is $d\mathbf{r}/dt_S dt_S/dt$, and $d\mathbf{r}/dt_S$ is \mathbf{v} , so the solution becomes $dt_S/dt = 1 - dt_S/dt \hat{\mathbf{r}}\cdot\mathbf{v}/c$. Solving for dt_S/dt ,

$$dt_S/dt = 1/(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c). \tag{5}$$

$d\mathbf{A}/dt$ is $d\mathbf{A}/dt_S dt_S/dt$, so Eqs 4 become

$$\begin{aligned}
d\mathbf{A}_{\text{LW}}/dt &= q\mathbf{a}/[cr(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2] \\
&\quad - q\mathbf{v}\mathbf{a}\cdot\hat{\mathbf{r}}/[c^2r(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^3] \\
&\quad + q\mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/[c^2r^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^3] \\
&\quad - q\mathbf{v}v^2/[c^2r^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^3] \\
&\quad - q\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/[cr^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2], \\
d\psi_{\text{LW}}/dt &= -q\mathbf{a}\cdot\hat{\mathbf{r}}/[cr(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^3] \\
&\quad - q\hat{\mathbf{r}}\cdot\mathbf{v}/[r^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2] \\
&\quad + q(\hat{\mathbf{r}}\cdot\mathbf{v})^2/[cr^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^3] \\
&\quad - qv^2/[cr^2(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^3].
\end{aligned} \tag{6}$$

The $1/r$ terms are radiative.

III. THE THOMAS TERMS

All but the simplest of retardation problems are intractable if an exact solution is attempted. The multivariate Taylor theorem² applies to other cases.

One way of performing a multivariate series expansion is with recursive applications of the Taylor theorem for one variable, but that results in terms that incomplete in their own order. The incomplete terms will occur in subsequent calculations in any case. Thus if the series expansion is to order a^1 and v^3 then the a^1v^3 terms are in the same order as the v^4 terms and must be dropped. Carrying the incomplete terms in intermediate calculations is inefficient but harmless. It is all right to selectively drop powers of some of the variables of the full

multivariate expansion in the final solution, so it is also all right to drop the same powers throughout. When the equations contain dt , it is normally only used for computing the derivatives, in which case it does not count as one of the variables of the multivariate expansion.

The Lorentz transform in vector form is

$$\begin{aligned}
\mathbf{r}' &= \gamma(\mathbf{r} - t\mathbf{v}) - (\gamma - 1)(\mathbf{r} - \mathbf{v}\mathbf{r}\cdot\mathbf{v}/v^2) \\
t' &= \gamma(t - \mathbf{r}\cdot\mathbf{v}/c^2),
\end{aligned}$$

with $\gamma = (1 - v^2/c^2)^{-1/2}$. When working in series form, the v^2 term in the denominator is cancelled by the $\gamma - 1$ term. That requires that the numerator be initially expanded to two more powers of velocity than will be needed. To order v^3 , the series solution is

$$\begin{aligned}
\mathbf{r}' &= \mathbf{r} - t[\mathbf{v} + v^2\mathbf{v}/(2c^2)] + \mathbf{v}\mathbf{r}\cdot\mathbf{v}/(2c^2) \\
t' &= t[1 + v^2/(2c^2)] - (\mathbf{r}\cdot\mathbf{v})/c^2 - v^2\mathbf{r}\cdot\mathbf{v}/(2c^4).
\end{aligned} \tag{7}$$

The calculations will be to order v^3 and a^1 . There are no a^2 terms in the solution for the first derivative.

The trajectory of the particle is

$$\begin{aligned}
\mathbf{r}_{1S} &= \mathbf{r} + \mathbf{v}(dt_{S1} + dt_{S2}) + 1/2 \mathbf{a}(dt_{S1} + dt_{S2})^2 \\
t_{1S} &= dt_{S1} + dt_{S2} - r/c.
\end{aligned} \tag{8}$$

The equation is a Taylor expansion of the particle's position, with the \mathbf{a} term not corresponding accurately to acceleration in a physical sense when the velocity is high. The location of the field point is

$$\begin{aligned}
\mathbf{r}_{1F} &= 0 \\
t_{1F} &= dt_{F1} + dt_{F2}.
\end{aligned}$$

$(dt_{S1} + dt_{S2})^2$ expands to $dt_{S1}^2 + 2dt_{S1}dt_{S2} + dt_{S2}^2$. The quadratic terms are in the same order as the $dt_{S1} dt_{S2}$ term, and they would normally have to be carried. For this particular calculation, the quadratic terms do not affect the final solution. In the interest of brevity, they will not be carried.

The particle must be on the light cone at time dt_{F1} , so dt_{S1} is not a free parameter. There is no requirement that it be on the light cone at time $dt_{F1} + dt_{F2}$, so dt_{F2} and dt_{S2} can be set to zero in solving for the light cone condition. The final solution would be the same if the light cone constraint were also imposed at time $dt_{F1} + dt_{F2}$.

The space difference, $\mathbf{r}_{1S} - \mathbf{r}_{1F}$, at time dt_{S1} is $\mathbf{r} + \mathbf{v} dt_{S1}$, and the magnitude of the vector is $r + \hat{\mathbf{r}}\cdot\mathbf{v} dt_{S1}$. The time difference, $t_{1S} - t_{1F}$, is $-r/c - dt_{F1} + dt_{S1}$, leading to the light cone condition $-r/c - \hat{\mathbf{r}}\cdot\mathbf{v} dt_{S1}/c = -r/c + dt_{S1} - dt_{F1}$. The equation evaluates to $dt_{S1} = dt_{F1}/(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)$. The solution could have been obtained in a simpler way from Eq 5. A more general approach is needed for other problems, and the method of successive approximation is usually required. The solution is expanded in a series in v then substituted into Eqs 8. Both ends of the vector are then transformed to the second frame of reference with the velocity \mathbf{v} .

$$\begin{aligned}
\mathbf{r}_{2S} &= \mathbf{r} + r\mathbf{v}/c + r\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/(2c^2) + rv^2\mathbf{v}/(2c^3) \\
&\quad + (dt_{F1}dt_{S2})[\mathbf{a} - \mathbf{a}\hat{\mathbf{r}}\cdot\mathbf{v}/c + \mathbf{a}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/c^2 \\
&\quad + \mathbf{v}\mathbf{a}\cdot\mathbf{v}/(2c^2)]
\end{aligned}$$

$$\begin{aligned}
t_{2S} = & -r/c - r\hat{\mathbf{r}}\cdot\mathbf{v}/c^2 - rv^2/(2c^3) - rv^2\hat{\mathbf{r}}\cdot\mathbf{v}/(2c^4) \\
& + dt_{F1}[1 - \hat{\mathbf{r}}\cdot\mathbf{v}/c - v^2/(2c^2) + (\hat{\mathbf{r}}\cdot\mathbf{v})^2/c^2 \\
& + v^2\hat{\mathbf{r}}\cdot\mathbf{v}/(2c^3) - (\hat{\mathbf{r}}\cdot\mathbf{v})^3/c^3] + dt_{S2}[1 - v^2/(2c^2)] \\
& + (dt_{F1}dt_{S2})[\mathbf{a}\cdot\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/c^3 - \mathbf{a}\cdot\mathbf{v}/c^2]
\end{aligned}$$

$$\mathbf{r}_{2F} = (dt_{F1} + dt_{F2})[-\mathbf{v} - v^2\mathbf{v}/(2c^2)]$$

$$t_{2F} = (dt_{F1} + dt_{F2})[1 + v^2/(2c^2)]$$

The velocity $\mathbf{v}_{23} = d\mathbf{r}/dt$ of the particle in the second frame of reference at time dt_{F1} is needed. $d\mathbf{r}$ is obtained by setting dt_{S2} to 0 in the space part of the solution, then subtracting the result from the full solution. The result is

$$d\mathbf{r} = dt_{F1}dt_{S2}[\mathbf{a} - \mathbf{a}\hat{\mathbf{r}}\cdot\mathbf{v}/c + \mathbf{a}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/c^2 + \mathbf{v}\mathbf{a}\cdot\mathbf{v}/(2c^2)].$$

Proceeding similarly for the time part of the solution leads to

$$dt = dt_{S2}[1 - v^2/(2c^2) + dt_{F1}\mathbf{a}\cdot\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/c^3 - dt_{F1}\mathbf{a}\cdot\mathbf{v}/c^2].$$

Then, in series form, $d\mathbf{r}/dt$ in the second frame of reference becomes

$$\begin{aligned}
\mathbf{v}_{23} = dt_{F1}[\mathbf{a} - \mathbf{a}\hat{\mathbf{r}}\cdot\mathbf{v}/c + \mathbf{a}v^2/(2c^2) + \mathbf{a}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/c^2 \\
+ \mathbf{v}\mathbf{a}\cdot\mathbf{v}/(2c^2) - \mathbf{a}v^2\hat{\mathbf{r}}\cdot\mathbf{v}/(2c^3)].
\end{aligned}$$

This velocity is used to transform to the third frame of reference. dt_{S2} was only needed for computing \mathbf{v}_{23} , so it is set to zero before performing the transform. The transform is an infinitesimal transform, so only the first power of velocity in Eqs 7 is needed. The solution is

$$\begin{aligned}
\mathbf{r}_{3S} = & \mathbf{r} + r\mathbf{v}/c + rv^2\mathbf{v}/(2c^3) + r\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/(2c^4) \\
& + dt_{F1}[\mathbf{a}r/c + \mathbf{a}rv^2/c^3 + r\mathbf{v}\mathbf{a}\cdot\mathbf{v}/(2c^3)] \\
t_{3S} = & -r/c - r\hat{\mathbf{r}}\cdot\mathbf{v}/c^2 - rv^2/(2c^3) - rv^2\hat{\mathbf{r}}\cdot\mathbf{v}/(2c^4) \\
& + dt_{F1}[1 - \hat{\mathbf{r}}\cdot\mathbf{v}/c - v^2/(2c^2) + v^2\hat{\mathbf{r}}\cdot\mathbf{v}/(2c^3) \\
& - (\hat{\mathbf{r}}\cdot\mathbf{v})^3/c^3 + (\hat{\mathbf{r}}\cdot\mathbf{v})^2/c^2 - r\mathbf{a}\cdot\hat{\mathbf{r}}/c^2 + r\mathbf{a}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{v}/c^3 \\
& - r\mathbf{a}\cdot\mathbf{v}/c^3 - rv^2\mathbf{a}\cdot\hat{\mathbf{r}}/(2c^4) - r\mathbf{a}\cdot\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/c^4]
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_{3F} = & (dt_{F1} + dt_{F2})[-\mathbf{v} - v^2\mathbf{v}/(2c^2)] + dt_{F1}dt_{F2}[\mathbf{a} \\
& - \mathbf{a} + \mathbf{a}\hat{\mathbf{r}}\cdot\mathbf{v}/c - \mathbf{a}v^2/c^2 - \mathbf{a}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/c^2 \\
& - \mathbf{v}\mathbf{a}\cdot\mathbf{v}/(2c^2)]
\end{aligned}$$

$$\begin{aligned}
t_{3F} = & (dt_{F1} + dt_{F2})[1 + v^2/(2c^2)] + dt_{F1}dt_{F2}[\mathbf{a}\cdot\mathbf{v}/c^2 \\
& - \mathbf{a}\cdot\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/c^3].
\end{aligned}$$

In this frame of reference the potential solution is just the static Coulomb solution, $\psi = q/r$. The solution can be viewed as being the solution for a constant velocity particle moving tangentially to the trajectory of the accelerated particle. In its frame of reference the static potential solution has existed forever, and the distant field point moves through it. The motion of the field point

does not result in a vector potential term in its frame of reference. The past and the future do not matter for light cone events, so the solution for an accelerated particle should be the same. The assumption is subject to further evaluation.

The scalar solution becomes $\psi = q/[(\mathbf{r}_{3S} - \mathbf{r}_{3F})\cdot(\mathbf{r}_{3S} - \mathbf{r}_{3F})]^{1/2}$, with $dt_{S2} = 0$, $dt_{F2} = 0$. In series form,

$$\begin{aligned}
\psi = & q/r - q\hat{\mathbf{r}}\cdot\mathbf{v}/(cr) - qv^2/(2c^2r) + q(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^2r) \quad (9) \\
& + qv^2\hat{\mathbf{r}}\cdot\mathbf{v}/(2c^3r) - q(\hat{\mathbf{r}}\cdot\mathbf{v})^3/(c^3r) + dt_{F1}[-q\hat{\mathbf{r}}\cdot\mathbf{v}/r^2 \\
& - q\mathbf{a}\cdot\hat{\mathbf{r}}/(cr) + qv^2\mathbf{a}\cdot\hat{\mathbf{r}}/(2c^3r) - 6q\mathbf{a}\cdot\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^3r) \\
& + 2q\mathbf{a}\cdot\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^3r) + 7/2qv^2\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r^2) \\
& - 6q(\hat{\mathbf{r}}\cdot\mathbf{v})^3/(c^2r^2) + 3q\mathbf{a}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r) - q\mathbf{a}\cdot\mathbf{v}/(c^2r) \\
& - qv^2/(cr^2) + 3q(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(cr^2)],
\end{aligned}$$

with $\mathbf{A} = 0$. The potentials must now be transformed back to the frame of reference of the field point. The potentials transform in the same way as the coordinates. The velocity of the field point at time dt_{F1} in the third frame of reference is obtained in the same way as in the above calculation for \mathbf{v}_{23} , except that dt_{F2} is used instead of dt_{S2} . The solution is

$$\begin{aligned}
\mathbf{v}_{31} = & -\mathbf{v} + dt_{F1}[-\mathbf{a} + \mathbf{a}\hat{\mathbf{r}}\cdot\mathbf{v}/c - \mathbf{a}v^2/(2c^2) \\
& - \mathbf{a}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/c^2 + \mathbf{v}\mathbf{a}\cdot\mathbf{v}/(2c^2) \\
& - \mathbf{a}v^2\hat{\mathbf{r}}\cdot\mathbf{v}/(2c^3) - \mathbf{v}\mathbf{a}\cdot\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/c^3].
\end{aligned}$$

After transforming the potentials with this velocity,

$$\begin{aligned}
\mathbf{A} = & q\mathbf{v}/(cr) + q\mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^3r) - q\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r) \quad (10) \\
& + q \, dt \, [\mathbf{a}/(cr) - 2\mathbf{a}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r) + \mathbf{a}v^2/(2c^3r) \\
& - \mathbf{v}\mathbf{a}\cdot\mathbf{v}/(2c^3r) + 3\mathbf{a}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^3r) - \mathbf{v}\mathbf{a}\cdot\hat{\mathbf{r}}/(c^2r) \\
& + 3\mathbf{v}\mathbf{a}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^3r) - v^2\mathbf{v}/(c^2r^2) - \mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/(cr^2) \\
& + 3\mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^2r^2)]
\end{aligned}$$

$$\begin{aligned}
\psi = & q/r - q\hat{\mathbf{r}}\cdot\mathbf{v}/(cr) + q(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^2r) - q(\hat{\mathbf{r}}\cdot\mathbf{v})^3/(c^3r) \\
& + q \, dt \, [-\mathbf{a}\cdot\hat{\mathbf{r}}/(cr) + 3\mathbf{a}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r) - 6\mathbf{a}\cdot\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^3r) \\
& - v^2/(cr^2) - \hat{\mathbf{r}}\cdot\mathbf{v}/r^2 + 3v^2\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r^2) + 3(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(cr^2) \\
& - 6(\hat{\mathbf{r}}\cdot\mathbf{v})^3/(c^2r^2)].
\end{aligned}$$

The subscript of dt_{F1} has been dropped in the solution. The vector equation has been multiplied by $1/c$ so that the powers of c in the retardation solutions will be the same as those of the Maxwell equations. It is of course possible to work in other systems of units. It is even possible to use a different system of units for the retarded potentials than is used for the fields, with the conversion factors being included in the equations for the fields.

This solution contains both the LW and the Thomas terms, with the full solution being the simple linear sum of the two. It will be helpful to separate the LW and the Thomas terms. The LW terms could be obtained by transforming to the frame of reference of the particle at time $t + dt$, computing the potentials as in Eqs 9, then

transforming them back with the negative of the same velocity. Another way of obtaining the same answer is to transform the potentials in Eqs 9 back in two steps, first with the velocity $-\mathbf{v}_{23}$, then with the velocity $-\mathbf{v}$. It is easier use Eqs 6 to extrapolate the exact LW solution to the time dt with the relationship $\mathbf{A}(dt) = \mathbf{A}(0) + d\mathbf{A}/dt dt$, and similarly for the scalar equation. After converting to series form,

$$\begin{aligned} \mathbf{A}_{\text{LW}} = & q\mathbf{v}/(cr) - q\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r) + q\mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^3r) \quad (11) \\ & + dt [-q\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/(cr^2) + 3q\mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^2r^2) \\ & - qv^2\mathbf{v}/(c^2r^2) + \mathbf{a}q/(cr) - 2\mathbf{a}q\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r) \\ & - q\mathbf{v}\mathbf{a}\cdot\hat{\mathbf{r}}/(c^2r) + 3\mathbf{a}q(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^3r) \\ & + 3q\mathbf{v}\mathbf{a}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^3r)], \\ \psi_{\text{LW}} = & q/r - q\hat{\mathbf{r}}\cdot\mathbf{v}/(cr) + q(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^2r) - q(\hat{\mathbf{r}}\cdot\mathbf{v})^3/(c^3r) \\ & + dt [-q\mathbf{a}\cdot\hat{\mathbf{r}}/(cr) + 3q\mathbf{a}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r) - 6q\mathbf{a}\cdot\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^3r) \\ & - q\hat{\mathbf{r}}\cdot\mathbf{v}/r^2 - qv^2/(cr^2) + 3q(\hat{\mathbf{r}}\cdot\mathbf{v})^2/(c^2r^2) \\ & + 3qv^2\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r^2) - 6q(\hat{\mathbf{r}}\cdot\mathbf{v})^3/(c^2r^2)]. \end{aligned}$$

The Thomas terms can now be segregated by subtracting Eqs 11 from Eqs 10.

$$\begin{aligned} d\mathbf{A}_T &= dt q\mathbf{a}\mathbf{v}^2/(2c^3r) - dt q\mathbf{v}\mathbf{a}\cdot\mathbf{v}/(2c^3r) \quad (12) \\ d\psi_T &= 0 \end{aligned}$$

By repeating the calculation with progressively higher powers of velocity, and observing how the terms evolve, it is not too difficult to infer the closed form solution. After dividing through by dt , it is

$$d\mathbf{A}_T/dt = q(\mathbf{a} - \hat{\mathbf{v}}\mathbf{a}\cdot\hat{\mathbf{v}})(\gamma - 1)/[(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2cr], \quad (13)$$

with $\gamma = (1 - v^2/c^2)^{-1/2}$. Unit velocity vectors are difficult to work with. If the expression for γ is substituted and the equation simplified they go away, but the equation is longer.

The solution was validated by expanding it in a series in \mathbf{a} and \mathbf{v} then comparing it to the derivation in series form. The two calculations were the same to order v^{30} , implying that the solution is exact. The existence of an exact solution indicates that the entire derivation could be performed in exact form, but some of the intermediate expressions are lengthy and difficult to simplify. The calculations have not been carried through.

The higher order velocity terms are probably not meaningful without also carrying the $\dot{\mathbf{a}}$ terms, but the compactness of equations that look like they are exact is nevertheless appealing.

A vector identity can be used to place the solution in the form

$$d\mathbf{A}_T/dt = q\hat{\mathbf{v}} \times (\mathbf{a} \times \hat{\mathbf{v}})(\gamma - 1)/[(1 + \hat{\mathbf{r}}\cdot\mathbf{v}/c)^2cr],$$

showing that, as expected, the Thomas terms vanish when the acceleration and velocity vectors are parallel or anti-parallel. The full retardation equation is the sum of either of these equations and Eqs 6.

The Thomas rotation is of order a^1v^1 , but the vector potential is of order v^1 , so the Thomas terms in potential form are of order a^1v^2 . When $\mathbf{a}\cdot\mathbf{v}$ is zero the ratio of the Thomas term in Eq 12 to the lowest order radiative Maxwell term in Eqs 6 is $v^2/(2c^2)$. The Thomas terms behave like a relativistic correction to the Maxwell terms in the far field.

Unless a way can be found to integrate the solution in a general way, it is necessary to first obtain the solution for the first time derivative, then integrate it. The integration is always easy for periodic solutions, however the static terms are lost. The static terms could be retained by directly retarding the E and B fields.

The Thomas terms are difficult to recognize in solutions, so it is often helpful to multiply the equation by T , carry it through the entire derivation, then set it to 1 in the last step. T sometimes drops out, meaning that the solution has reduced to the LW result.

The magnitude squared of the total vector potential can be obtained by multiplying Eq 13 by T , adding the result to the LW terms in Eqs 6, then computing $\mathbf{A}\cdot\mathbf{A}$. To order v^3 , the result is

$$\begin{aligned} \mathbf{A}\cdot\mathbf{A} = & q^2v^2/(c^2r^2) - 2q^2v^2\hat{\mathbf{r}}\cdot\mathbf{v}/(c^3r^2) \\ & + dt [-2q^2v^2\hat{\mathbf{r}}\cdot\mathbf{v}/(c^2r^3) + 2q^2\mathbf{a}\cdot\mathbf{v}/(c^2r^2) \\ & - 2q^2v^2\mathbf{a}\cdot\hat{\mathbf{r}}/(c^3r^2) - 6q^2\mathbf{a}\cdot\mathbf{v}\hat{\mathbf{r}}\cdot\mathbf{v}/(c^3r^2)]. \end{aligned}$$

T has dropped out of the solution, showing that the magnitude of the vector is the same as the LW value. That means that the full solution is the LW solution, followed by an infinitesimal space rotation. Carrying more powers of velocity in the calculation does not affect the conclusion. A space rotation does not affect the scalar potential, so it is the same as for the LW solution.

When applying the Lorentz transform, a space rotation in the second frame of reference, no matter how large, leaves the invariant quantity $r^2 - c^2t^2$ unaltered. The invariant quantity does not directly apply to the 4-potential, but since it transforms in the same way as the coordinates, and since it would be possible to work in a system of units where the vector potential has the units of distance and the scalar potential has the units of time, it is likely that mathematical inconsistencies would arise if the perturbation of the LW vector potential were more than a space rotation.

The equations look like linear equations, however a^2 terms appear in the second derivatives (not shown), tending to obscure the meaning of linearity. Extrapolations around a circle can be built up as a series of infinitesimal rotations, each of which is linear, but the overall relationships of a circle are not linear. As applied to the retardation equations, each additional infinitesimal rotation requires another differentiation, so the contravariant tensors can be viewed as achieving linearization by differentiation, implying that they are not impaired when compared to nonlinear representations if the rank of the tensor is sufficient. The contravariant tensor of the second rank represents the first derivatives⁴, which are not

very impressive in their capabilities.

The \dot{a} terms drop out of the solution for the first derivative, but not in the solution for the second derivative. Other \dot{a} terms would appear if the retardation solution for the first derivative were differentiated exclusively in the first frame of reference. However, from the perspective of the observer in the other frame of reference, they would not correctly represent the angular relationships of the system. That does not mean that retardation solutions should not be differentiated in the first frame of reference. It means that the derivatives obtained that way are incomplete or inconsistent. The argument can be applied recursively, so there is no alternative to using incomplete or inconsistent retardation equations if they are linear. (Gödel's proof does not include the qualification to linear equations. See www.wikipedia.org.)

Space rotations are the basis of the tensor series, and each tensor to at least the fifth rank is irreducible⁴, implying that mathematically complete retardation equations do not exist. It could be that real space-time is more degenerate than the tensor series, but probably not. However, mathematical degeneration might occur at some point if the dimensionality of the problem is restricted to four, just as the \dot{a} terms are degenerate in the Newton equations. The Newton equations represent three copies of one space coordinate and one time coordinate. In being vector equations, that is still true in Minkowski space. These relationships suggest that when fully extended to three space dimensions (with the third derivative and the tensor of the fourth rank), the retardation equations will be degenerate in \ddot{a} .

Degeneration in \ddot{a} would imply that the tensor of the fourth rank plays a special role in the linear relationships of the four dimensional space, with the equations taking on a new completeness and consistency in that order. It is believable that the geometry of the four dimensional space, linear or otherwise, is of a finite complexity, and that it can be comprehensively represented. That would be the end of the quest unless the dimensionality of real space-time is greater than four.

The tensor irreducibility theorem only applies to linear equations. Its basis is that the multipole order increases with the rank of the tensor, and multipoles cannot be synthesized from linear combinations of lower order multipoles.

IV. SOME FIELD EQUATIONS

The Proca equations are⁶

$$-\nabla^2 \mathbf{A} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\nabla L - \alpha^2 \mathbf{A}$$

$$-\nabla^2 \psi + \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{1}{c} \frac{\partial L}{\partial t} - \alpha^2 \psi$$

$$L = \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \psi}{\partial t}$$

The scalar L is known as the Lorentz condition. After applying the vector identity $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$, the equations become

$$\nabla \times \nabla \times \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\nabla L - \alpha^2 \mathbf{A} \quad (14)$$

$$-\nabla^2 \psi + \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{1}{c} \frac{\partial L}{\partial t} - \alpha^2 \psi.$$

The terms on the left are not the Maxwell equations. However, the solutions of the LW equations are in the Lorentz gauge, and in that gauge L is zero, so when working with those equations the $\nabla \cdot \mathbf{A}$ term in the left part of the vector equation can be replaced with $-(\partial \psi / \partial t) / c$. Similarly, in the scalar equation $+(\partial^2 \psi / \partial t^2) / c^2$ can be replaced by $-(\partial / \partial t)(\nabla \cdot \mathbf{A}) / c$. The equations are now

$$\nabla \times \nabla \times \mathbf{A} + \nabla(\partial \psi / \partial t) / c + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\nabla L - \alpha^2 \mathbf{A}$$

$$-\nabla^2 \psi - (\partial / \partial t)(\nabla \cdot \mathbf{A}) / c = \frac{1}{c} \frac{\partial L}{\partial t} - \alpha^2 \psi.$$

In Eq 14, the $\nabla(\nabla \cdot \mathbf{A})$ term on the left cancels the same quantity in ∇L when ∇L is expanded. There is no static spherical solution. Since L is zero, it is all right to invert its sign, and doing so results in the static solution in Eq 21. The $\nabla(\nabla \cdot \mathbf{A})$ term ceased to be a static term when the substitution $\nabla \cdot \mathbf{A} = -(\partial \psi / \partial t) / c$ was made, so inverting the sign of L in the unmodified Proca equations does not have the same effect. The restructured equations differ from the Proca equations by more than a sign change in static solutions.

Inverting the sign of L in the scalar equation does not affect the static scalar solution. After inverting the sign of L in both equations, then inverting all the signs of the scalar equation, the final solution becomes

$$\nabla \times \nabla \times \mathbf{A} + \frac{1}{c} \nabla \frac{\partial \psi}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = \nabla L - \alpha^2 \mathbf{A} \quad (15)$$

$$\nabla^2 \psi + \frac{1}{c} \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \frac{1}{c} \frac{\partial L}{\partial t} + \alpha^2 \psi. \quad (16)$$

The scalar and vector potentials can be viewed as having the units of distance in these equations, but with the viewpoint not being unique.

It follows from the method of derivation that these equations are equivalent to the Proca equations when the solutions are in the Lorentz gauge, except that the Proca equations do not contain a static vector solution. The equations are not equivalent to the Proca equations when the solutions are not in the Lorentz gauge. Equations that are similar to these are derived in the supplemental material at www.s-4.com/tensor, indicating that the restructuring is not arbitrary.

The terms on the left are the Maxwell equations, and the terms on the right (with $\alpha = 0$) are zero when L is zero. In this particular gauge, the equations reduce to

the Maxwell equations. The equations are therefore fully compatible with the LW equations, even though they are somewhat more general than the Maxwell equations.

The Lorentz condition is not zero when the solutions contain Thomas terms. The r_0^3 solutions of the following section are solutions to these equations (with $\alpha = 0$) but they are not solutions to the Maxwell equations or the Proca equations. No arguments are presented that the retardation solutions should be solutions to these equations. It was simply noticed that they are – for low order solutions only.

From Eq 3, $\nabla \cdot \mathbf{H} = \nabla \cdot (\nabla \times \mathbf{A})$. The right side of the equation is identically zero, leading to Eq 17, which is one of the Maxwell equations.

Reversing the order of differentiation of the second term on the left side of Eq 16 then factoring the left side leads to $\nabla \cdot (\nabla \psi + (\partial \mathbf{A} / \partial t) / c)$. From Eq 2, the term can be written as $-\nabla \cdot \mathbf{E}$. Including the other terms in the scalar equation provides Eq 18. The equation reduces to one of the Maxwell equations when L and α are zero. The equation does not lead to the lack of charge conservation if the virtual charge in one region is canceled by virtual charge of the opposite sign in another region. It has not been determined if the solutions do globally conserve charge.

From Eq 2, $\nabla \times \mathbf{E} = \nabla \times (-\nabla \psi - (\partial \mathbf{A} / \partial t) / c)$. $\nabla \times (\nabla \psi)$ is identically zero. Reversing the order of differentiation of the $\nabla \times (\partial \mathbf{A} / \partial t) / c$ term and substituting from Eq 3 provides Eq 19, which is one of the Maxwell equations.

In Eq 15, $\nabla \times \nabla \times \mathbf{A}$ is $\nabla \times \mathbf{H}$. Reversing the order of differentiation in the other two terms on the left side of the equation and substituting from Eq 2 leads to Eq 20. The equation reduces to one of the Maxwell equations when L and α are zero.

$$\nabla \cdot \mathbf{H} = 0 \quad (17)$$

$$\nabla \cdot \mathbf{E} = -(\partial L / \partial t) / c - \alpha^2 \psi \quad (18)$$

$$\nabla \times \mathbf{E} + (\partial \mathbf{H} / \partial t) / c = 0 \quad (19)$$

$$\nabla \times \mathbf{H} - (\partial \mathbf{E} / \partial t) / c = \nabla L - \alpha^2 \mathbf{A} \quad (20)$$

The static solutions could be obtained by directly retarding the E and B fields. The scalar L must also be retarded in order to check the solutions for computational errors. There is a restriction to low order solutions. α can usually be set to zero in solutions for the local region of the cosmos, making it possible to validate the retardation solutions without knowing the undifferentiated scalar and vector potentials.

Suppose that α is $1/r_r$, where r_r is the range of the fields. The range of the fields is not currently known. Then, in spherical coordinates, $\psi = q \exp(-r/r_r)/r$, $\mathbf{A} = 0$ is a static solution to Eqs 15 and 16, with $\nabla^2 \psi = \alpha^2 \psi$. When $r \ll r_r$ $\exp(-r/r_r)$ approaches 1 and the potential solution approaches $\psi = q/r$. (The units and scaling relationships of α depend on the system of units utilized. Alternatively, if the potentials are

appropriately scaled with the units of distance then α is explicitly $1/r_r$, and the equations shown are of this form. That entails including a cosmological constant in local equations. That may seem wrong at first, but why should we be different than the rest of the universe?)

Another static solution is

$$A_r = -k^2 r_r \exp(-r/r_r) / r^2 - k^2 \exp(-r/r_r) / r, \quad (21)$$

with $\psi = 0$, $A_\theta = 0$ and $A_\phi = 0$. k^2 has the units of distance squared and represents the source strength. The source strength can also be represented as br_r , which is more appropriate in a physical sense. The Schwarzschild radius is proportional to mass, making it possible to represent mass with the units of distance, but with the scaling relationships not directly applying to non-metrical equations. (This equation is analytically awkward to work with. Simply substituting $r = d r_r$, where d is a number $\ll 1$, is an expedient way of evaluating its behavior.)

$\nabla \times \mathbf{A}$ is 0, and $\nabla^2 \mathbf{A} = \alpha^2 \mathbf{A}$. The solution is static, so the scalar equation is also satisfied. When $r \ll r_r$, A_r is $\approx -k^2 r_r / r^2$ and $\nabla \cdot \mathbf{A}$ is $\approx k^2 / (r_r r)$. The first derivative of the potential decays with distance at a lower rate than the potential, which is only possible with exponential equations. The second derivative, $\nabla(\nabla \cdot \mathbf{A})$, decays approximately as $-k^2 / (r_r r^2)$, which represents an inverse square law vector field in the local region.

The electrostatic and gravitational fields are the only known inverse square law fields, but it has not been determined if the solutions are meaningful.

The static scalar and vector solutions define two source terms that could be retarded, which would have the effect of assimilating $\alpha = 1/r_r$ into the retardation equations. But $-k^2 / (r_r r^2)$ simplifies to $-b/r^2$ if the source strength is written as br_r , making it possible to scale local solutions without knowing r_r . The scaling relationships follow from Newtonian gravity in the local region, without cosmological complications.

The exponential expressions containing r_r must nevertheless be carried in the differentiations, since r_r does not drop out until after the first differentiation. The net effect is that the cosmological influence, though present, is hidden from us in the static solutions of this order in the nearby region.

A peek at the solutions for the tensor of the fourth rank can be gained by simply differentiating again.

$$\nabla \cdot (\nabla(\nabla \cdot \mathbf{A})) = \nabla \cdot \mathbf{a} = k^2 \exp(-r/r_r) / (r r r^3)$$

\mathbf{a} is the acceleration. $\nabla \cdot \mathbf{a}$ is zero in Newtonian gravity.

The third derivative of the static scalar solution is

$$\nabla(\nabla \cdot (\nabla \psi)) = -\nabla(\nabla \cdot \mathbf{E}) = -q \exp(-r/r_r) / (r^2 r_r^2 + r r_r^3)$$

As discussed above, the scalar and vector potentials in these solutions are in a cosmological system of units. The MKS scaling relationships follow from Newtonian gravity, with k^2 becoming $r_r G m$. The MKS electrical scaling

relationships follow from the equation for the E field of a charged particle, with $q_{\text{mks}} = q/(4\pi\epsilon_0)$. The cosmological influence is of first order in the infinitesimal, but it is much too small to detect in the static solutions for the nearby region. Its magnitude will need re-evaluation after the dynamic solutions are obtained.

A term that is first order in the infinitesimal, no matter how small, cannot be neglected when integrating to cosmological distances.

The unmodified Proca equations contain a similar scalar solution. In a quantum context the α of the Proca equations is mc/\hbar , and with the units of α depending on the system of units used.

The differentiations cannot be performed exclusively in the first frame of reference unless the solution is static. The Thomas corrections are required.

There are indications that Eqs 15 are only valid for the first derivatives of the electrical solutions, and it would be surprising if the same equations are valid for the second derivatives of the radiative gravitational solutions, even though the static solution is satisfied. Although undesirable, it is possible to proceed with the development of retardation equations without yet knowing the associated field equations. Matching the retardation equations with field equations will eventually be essential, but it cannot be done until the radiative gravitational solutions are obtained.

The radiative solutions to the differential equations will also be needed, and there could easily be inconsistencies between the two solutions, inconsistencies that would eventually have to be resolved. Furthermore, despite the inferences that follow from currently popular theories, the differential equations of some order might contain the cosmological redshift, which could require a reformulation of the retardation problem.

The second derivative gravitational solutions for the local region are likely to fare better, and they are the appropriate starting point. But even in the local region, neglecting the third derivatives causes the solutions to be approximations. The relationships are fundamentally nonlinear, and obtaining exact solutions with linear equations is not possible. It remains to be seen if the solutions apply to physical problems.

Since there are no absolute points of reference, the locally observable relationships of an acceleration wave do not follow directly from the retardation solutions. Two nearby points have to be compared. The acceleration wave behaves more like a potential than a field. Accelerated observers are the only ones that are physically realizable in the acceleration wave, so the locally measurable relationships have to be evaluated in their frame of reference. Even from the perspective of a distant observer, Newtonian inferences should not be applied without further investigation, because the wave does behave more like a potential than a field, and it is not included in the Newton equations. The behavior of the acceleration vector should be Newtonian when the radiative terms can be neglected.

The radiative solutions of the general theory of relativity are obtained in a space that is asymptotically flat. The above relationships cannot exist in a space where r_r is infinite. There may be a possibility of mathematical inconsistencies occurring between the two calculations when the radius of a radiative solution in spherical coordinates is taken as being infinite. Although unintuitive, mathematical singularities at infinity do occur, and they are capable of causing the form of a solution to change abruptly in the limit. The solution obtained by prematurely setting $r_r = \infty$ in Eq 21 is not an approximation. The derivatives are totally wrong.

Curvature relationships are especially susceptible to odd behavior at infinity, because the radius of curvature is also infinite in the limit. It is possible for the calculations to contain unnoticed but undefined ∞/∞ terms. In the Newtonian approximation, the acceleration in the static GR solution is the gradient of the expansion factor. In containing nothing more than a quadrupole, the expansion factor is zero in the radiative far field. It is important that we know for sure that it is zero in the far field, as one of the consequences of a non-zero value would be that a collapsing spherically symmetric mass would radiate. Another reason for being sure is that if the GR expansion factor is in fact both zero and well behaved in the limit then there will be grave doubts of the validity of the retardation equations.

V. A SIMPLE ANTENNA

The calculations of this section attempt to integrate around a circle with the \mathbf{a} terms only. That can be done perfectly with the Newton equations, because they are degenerate in $\dot{\mathbf{a}}$. Retardation equations are probably not degenerate in $\dot{\mathbf{a}}$, but the problem can be developed as a series expansion. (The analogy leaves out some steps, but it may have intuitive merit.)

The calculations are to order r_0^3 , where r_0^3 is the radius of the orbit for a single charged particle. The calculation fails in the r_0^4 solution. The r_0^4 terms look like low order terms, and they are in some sense, but it is also true that the angular velocities are higher in the near field region. The Thomas precession is a rotation, so angular relationships are important. The particle velocity cannot exceed c , but there is no upper bound to the angular velocities. That might cause the neglected $\dot{\mathbf{a}}$ terms to become important in the near field region. The same relationship can cause small systems to behave differently than their larger counterparts. The LW equations do not fail in the near field region, but they are insensitive to angular relationships.

The calculations are straight forward but much too lengthy to show here. They are shown in the supplemental online material at www.s-4.com/som1. The material is in the form of raw computer output and it is not very readable.

After computing the fields, converting the solution

back to the Cartesian system, then setting the x and y coordinates to zero, the far field Thomas solution along the z axis is

$$\begin{aligned} E_x &= qr_0^3\omega^4 \cos(\omega t - \omega z/c)/(2c^4z) \\ E_y &= qr_0^3\omega^4 \sin(\omega t - \omega z/c)/(2c^4z) \\ E_z &= 0 \\ H_x &= -qr_0^3\omega^4 \sin(\omega t - \omega z/c)/(2c^4z) \\ H_y &= qr_0^3\omega^4 \cos(\omega t - \omega z/c)/(2c^4z) \\ H_z &= 0. \end{aligned}$$

The orbit is in the x-y plane. The LW terms in the solution are of the same form, except that they are multiplied by $r_0\omega^2/c^2$ rather than $r_0^3\omega^4/(2c^4)$. $r_0\omega$ is v , so the ratio is $v^2/(2c^2)$.

Both the Thomas and the LW terms represent circularly polarized Maxwellian radiation. The Maxwell equations constrain the fields without specifying what causes them, so all of their relationships are applicable to unconventional systems when the solutions satisfy the equations.

Textbooks sometimes attribute the physical basis of solutions to the Maxwell equations when the actual basis is the LW equations. The multipole solutions of field equations are not capable of specifying the physical properties of the source, so the source terms that the Maxwell equations can accommodate are more general than most presentations indicate. The source terms do not result in multipole terms that are not already known. It is rather that the physical basis of the solution can be unfamiliar. Unambiguous interpretation of the solutions of field equations is not possible without retardation equations. Conversely, because of the ambiguities of multipole solutions, the Maxwell equations are more general in a physical sense than was thought at first. There are other near field terms in the solution along the z axis that are not a solution to the Maxwell equations.

The far field Thomas terms along the x axis are

$$\begin{aligned} E_x &= qr_0^3\omega^4 \cos(\omega t - \omega x/c)/(2c^4x) \\ E_y &= qr_0^3\omega^4 \sin(\omega t - \omega x/c)/(2c^4x) \\ E_z &= 0 \\ H_x &= 0 \\ H_y &= 0 \\ H_z &= qr_0^3\omega^4 \sin(\omega t - \omega x/c)/(2c^4x) \end{aligned}$$

The \mathbf{E} vector rotates in the x-y plane with a constant magnitude, while the \mathbf{H} vector is parallel to the z axis. The solution can be decomposed into two parts. One of them is equivalent to an appropriately scaled Maxwellian dipole at the origin and parallel to the y axis. After subtracting this component, the residual is the lone E_x component, which is parallel to the direction of propagation. Even though the other terms are Thomas terms, only the residual will exhibit any physical behavior that is not contained in the Maxwell equations.

While the Thomas component of the \mathbf{E} field that is parallel to the propagation direction can be mathematically separated, it is of order v^2/c^2 times the Maxwellian \mathbf{E} field in the same solution, and it probably cannot be physically separated from the Maxwell terms. With this interpretation, the component that is parallel to the propagation direction does not represent a separate form of radiation, but is rather a minor relativistic correction to the Maxwellian wave.

The gradient of the Lorentz condition defines a third vector in this solution, but it is not dimensionally consistent with the Thomas electrical components, so it cannot be combined with any of them in the same way that the \mathbf{E} and \mathbf{B} fields are combined to synthesize a separate form of radiation. The second rank tensor represents the first derivatives. The third rank tensor is more appropriate for the analysis of the second derivatives of ∇L .

The tensor decomposition equation⁴ was used to obtain the decomposition products of the third rank tensor. The calculations were performed by extrapolating the potentials in space and time to obtain the second rank tensor. The decomposition equation is in 3-space, so the 3+1 space of the first frame of reference is appropriate for the calculation. A vector extrapolates as three scalars in 3+1 space. The second rank tensor has a 3x3 structure in 3+1 space, and the third rank tensor is its gradient. The calculations are shown at www.s-4.com/tensor. The calculations were performed in an anisotropic space of an assumed form. They could be performed in a more conventional space, although some interpretation of the behavior of the $\partial\psi/\partial t$ terms may be necessary in a conventional space. The calculations are easily converted to the 4-vector form.

The three vectors and the scalar are

$$\begin{aligned} \nabla[\nabla \cdot \mathbf{A} + (\partial\psi/\partial t)/c] &= \nabla L \\ (\partial/\partial t)(\nabla \times \mathbf{A}) &= \partial \mathbf{H}/\partial t \\ \nabla^2 \mathbf{A} - \nabla(\partial\psi/\partial t)/c &= -\nabla \times \mathbf{H} + \nabla(\nabla \cdot \mathbf{A}) \\ &\quad - \nabla(\partial\psi/\partial t)/c \\ \nabla \cdot [\nabla \psi + (\partial \mathbf{A}/\partial t)/c] &= -\nabla \cdot \mathbf{E}. \end{aligned} \tag{22}$$

The other decomposition products are two quadrupoles and an octupole. The symmetries of the third rank tensor indicate that its 4-potential radiative solutions will be much richer than those of the second rank tensor. Vector and 4-vector equations are not necessarily impaired if they are obtained from a tensor of sufficient rank.

As discussed in the online material, calculations in 3+1 space contain terms that differ by a factor of 3 from their 4-space equivalents. The terms are symmetric, and 4-vector equations do not have symmetric terms, causing the connection to 4-vector equations to be nebulous. In not having a direct connection to 4-vector equations, and not affecting the Maxwell equations, the factor of 3 does not appear in the literature. The tentative conclusion is that the factor of 3 can be dropped, and it has been dropped in the decomposition products. There are indications that the factor could be carried, and that there

is nothing fundamentally wrong with it, but carrying it would upset many familiar equations.

After performing two consecutive infinitesimal transforms in the frame of reference of the particle then integrating the second derivative twice (not shown), the solution is not a solution to the field equations. That is to be expected, since third order field equations should be required in the next order. (The Thomas precession vanishes if the r_0^3 terms are not carried. Similarly, the r_0^4 terms are required for a minimal representation of the second derivative.)

Differentiating again in the frame of reference of the particle then integrating again in the frame of reference of the field point results in a coupling between the orders. If the r_0^4 terms are dropped in the second derivative solution then the remaining terms are different from the solution for the first derivative, but with the only difference being that all of the Thomas terms are multiplied by 2. Since the full solution is the linear sum of the LW and Thomas terms, the solution is also a solution to the field equations. The coefficients for the Thomas terms of L for the 1st through 8th derivatives, after multiple integrations, are approximately 0.5, 1.0, 1.6, 2.1, 2.7, 3.3, 3.9, 4.4.

The series shows that elevating the rank of the associated tensor by one will have a substantial effect on the magnitude of the Thomas terms. From a different perspective, the same calculation shows that the influence of tensors of even very high rank can be folded into the solution for the first derivative, making it unnecessary to obtain the complete solution for the higher rank tensor in order to obtain better accuracy for the first derivative. It seems that there should be a better way of obtaining an accurate solution for the first derivative.

It is indicated that the coefficients of the Thomas terms of the above solutions should be much larger than the values shown, but with the form of the first derivative solutions not being perturbed by tensors of higher rank. It could be that, in a more complete development, the tensor series that is associated with the retardation series will be fully orthogonal, with none of the first derivative terms being perturbed by tensors of higher rank. The tensor series would be much better behaved that way, and the possibility is still under investigation.

VI. LABORATORY EVALUATION

The velocity of conduction electrons in stationary copper wire is so low that the Thomas terms are undetectable in those configurations.

When the system is mechanically rotating and excited with alternating current, each conduction electron must be paired with a proton, and the total charge is enormous. That causes the magnetic fields due to the proton and electron currents to be separately enormous. However, the LW solution for a rotating current loop is the same as for a stationary loop when the electrons and

protons are retarded separately, so the enormity of each of the two cancelling fields is not detectable. The additional relativistic corrections associated with the Thomas terms do cause the solution to depend on the mechanical angular velocity.

Even with rotating equipment, the Thomas terms are very weak, and it may not be feasible to detect them unless a way can be found to separate them from the Maxwell terms. It could be that adequate separation is not achievable for the first derivative solutions. In that case the following material may be of interest for higher order solutions.

The solution for a current loop excited by alternating current is unaltered if the loop is physically rotated, making it impossible for terms that are first order in the mechanical angular velocity to appear in the solution, although there are Thomas terms that are quadratic in velocity. Discovering a configuration that does have first order terms would be highly desirable.

Integrating the first time derivative of the retarded potentials is not well suited to the analysis of the low frequency near field terms in such solutions. It will be better to directly retard the E and B fields in order to obtain the static and quasi-static fields. The quasi-static near field terms have not yet been investigated, and it may be worthwhile to do so.

As Eqs 12 show, the $\nabla\psi$ terms makes no contribution to the Thomas E field. There are also no $\nabla\psi$ terms in the E field near a magnetic toroid that is excited with alternating current, and a short dipole sensor will not respond to it, even though it is Maxwellian. Similar and well understood experimental complications are to be expected of the non-radiative near field terms of the Thomas E field. A Maxwellian detector might not respond in the expected way unless the term is an approximate solution to the Maxwell equations. The complication should not arise when the solution is predominately just one quasi-static field, which will frequently be the case in the near field.

As discussed in Section V, the radiative first derivative Thomas terms do not appear to be physically separable from the Maxwell terms, making them undetectable in practice except under extreme conditions. The decomposition products of the third rank tensor in Eqs 22 suggest that separation will be easy for the second derivative solutions. However, Eq 21 and its associated equations indicate that the ∇L decomposition product is not an electrical field, so it may be better to continue on to the tensor of the fourth rank, which constrains the relationships among six vectors⁴.

It is not necessary to know how to build a receiving antenna for non-Maxwellian radiation in order to detect it. It is sufficient to build two transmitting antennas then measure the interaction energy. The interaction energy can be computed by integrating the sum of the two fields over a sphere at infinity. When the antennas are operated at slightly different frequencies, the interaction energy will be manifested as a modulation of the current

flow in each antenna. A similar approach may be useful for near field terms when the available detectors are not satisfactory.

It is plausible that there are electrical fields associated with an accelerated or jerked mass, although that will not be known until the retardation solutions are obtained. They might provide a readily accessible desktop method of laboratory evaluation. Since the field is the second derivative of the potentials, the terms of experimental interest may be in the order \dot{a} . A hammer might be more effective than a flywheel.

Conduction electrons behave like a fluid with mass when a metal is accelerated, causing the metal to acquire a weak electrostatic dipole moment. Abrupt changes in the acceleration are probably capable of producing a magnetic field. In the Barnett effect³, a rotating mass acquires a weak and sustained magnetic field, which is attributed to the spin of the electron. Weak piezoelectric effects may also need consideration in some configurations. It will obviously be important to not confuse known effects with the retardation solutions.

Inferring the electrical scaling relationships of gravitational solutions will require some plausible guesses, since the two fields are perfectly orthogonal in lower order representations. The connection between the br_r source term in Eq 21 and the Schwarzschild radius, when considered in relation to the Dirac field limits, provides a plausible method of inferring the scaling. The connection is that the Schwarzschild radius and the Dirac limits both represent limiting conditions, and the electrical and gravitational fields are probably on an equal footing in the limit. The calculation requires a pseudo-metrical interpretation of the retardation equations. The Dirac field limits are not sharply defined, and a pseudo-metrical approximation will not be very accurate, but order of magnitude estimates are adequate for designing laboratory equipment. The vector potential points inward in the gravitational solution. When represented as a displacement in space, it cannot have a magnitude greater than the radius of the mass, which can be the basis of a rough scaling relationship. These relationships are developed further in Ref. 7. The material at www.s-4.com/pulsar is more current and more complete.

The scaling relationships of the electrical solutions of the general theory of relativity have no observational

confirmation and they are not commensurate with the Dirac field limits, so they will not be useful in developing the coupling between the electrostatic and gravitational fields in dynamic systems. There is probably no coupling in static systems, just as there is no coupling between the E and B fields in static solutions. Classical calculations cannot be commensurate with quantum results until the Planck constant is assimilated.

But by of the identity $\alpha = \mu_0 c q_e^2 / (2h)$ (MKS units), assimilating the fine structure constant is equivalent to assimilating the Planck constant. The fine structure constant is a pure number quantity. We need a clear and precise understanding of why the geometry of the four dimensional space contains a ratio that is this particular value. Until we do, α can be used empirically, along with the retardation equations, in developing the scaling relationships of the coupling of the electrostatic and gravitational fields. The empirical relationships will eventually be useful in inferring the essential meaning of the constant. The numerical value of the constant should then be computable. In being purely geometrical, the solution should be exact *if* the dimensionality of real space-time is four.

It has been established by the equivalence principle and its consequences that acceleration and gravity are not distinguishable with the second derivative. It is therefore possible for a physical constant that was originally discovered in an electrical context to have a meaning that cannot be inferred in an electrical context.

There are pitfalls in relying on empirical geometry, with the worst of them being a loss of meaning rather than a loss of numerical accuracy.

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